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INTERNAL MEMORANDUM

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TOM TIDDLER'S GAME

by

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The author wishes to express his gratitude to his wife, who has played Tom Tiddler's game as a child, for informing the author of the existence of the game and for her observation that children instinctively play the strategies we have outlined.

I'm on old Tom Tiddler's ground
Picking up gold and silver*

ABSTRACT

A number of attackers try to reach a goal guarded by several goalies. The goalies and attackers approach each other with fixed velocity but can maneuver transversely within a specified velocity range. The largest miss distance between an attacker and the nearest goalie at the termination of the game is determined for an arbitrary initial deployment of the goalies and the attackers. Optimal strategies and optimal initial placements for the defense and attack are determined. The problem may be generalized so that the constraints are on the *m*th time derivative of the position vector. For the free evasion problem a simple transformation reduces the problem to one involving velocity constraints.

*With this chant English children begin a game very similar to the game analyzed in this paper. Tom Tiddler plays the role of the goalies and the trespassers are the attackers. After the first onslaught any trespassers caught must assist Tom Tiddler during the next play of the game. See Ref. 1.

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I. INTRODUCTION

In a previous paper[2] the author discussed an evasion problem involving one attacker evading an arbitrary number of goalies. In this paper it is shown that the generalized problem in which the number of attackers is arbitrary can also be solved analytically.

The game we consider is played in two dimensions. At the start of the game the n goalies are all on a line facing m attackers arrayed along a parallel line some distance away. The lines of goalies and attackers approach one another with a fixed relative velocity. The game ends when the line of attackers meets the line of goalies along a line we shall call the intercept line. As the two groups approach one another each participant maneuvers from side to side within certain specified constraints. The attackers perform these maneuvers with the object of avoiding the goalies. The goalies on the other hand try to catch the attackers. The success of the attack is measured by the distance between the most successful of the attackers and the goalie nearest him at the end of the game.

The problem may be formulated in terms of the projection of the jth goalie's position on the intercept line as a function of time. We denote this quantity by $x_j(t)$. The corresponding variable for the kth attacker is denoted by $y_k(t)$. For the purpose of the following discussion we impose the following constraints on the transverse velocities:*

$$|\dot{y}_k(t)| \leq V_A \tag{1}$$

and

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*In this paper we consider only the free evasion problem. For velocity constraints it is not difficult to generalize our results to the case when the attackers must have access to a goal region beyond the intercept line[2]. For the free evasion problem it is not difficult to generalize the results in the present paper to the case when the constraints given by Eqs. (1) and (2) are replaced by limitations on $(d/dt)^{\ell}y_j$ and $(d/dt)^{\ell}x_k$ (see appendix to Ref. 2).

$$|\dot{x}_y(t)| \leq V_G < V_A \tag{2}$$

where the dot indicates differentiation with respect to time.* The score of the game is formally given by

$$\begin{aligned} S = \mathrm{MAX} \Big| & \mathrm{MIN}\Big| |y_1(T) - x_1(T)|,\ |y_1(T) - x_2(T)|, \ldots \Big|, \\ & \mathrm{MIN}\Big| |y_2(T) - x_1(T)|,\ |y_2(T) - x_2(T)|, \ldots \Big|, \ldots, \\ & \mathrm{MIN}\Big| |y_m(T) - x_1(T)|, \ldots, |y_m(T) - x_n(T)| \Big| \Big| \quad . \end{aligned} \tag{3}$$

We shall obtain a formula which determines S for an arbitrary set of initial values $x_1(0), x_2(0), \ldots, y_1(0), y_2(0), \ldots$ and a specified termination time T, assuming that the goalies maneuver in the best possible way to minimize S and that the attackers perform optimally in their desire to maximize S.

The result is obtained in several easy stages. In the next section we briefly review the results for one attacker against n goalies.

In Sec. III a particular configuration is considered in which the initial deployment of both the goalies and attackers is optimal. By making a guess at the optimal attack strategy we obtain a lower bound on the score S for this configuration.

In Sec. IV the basic defense strategy is discussed.

In Sec. V an algorithm for defense and offense strategy is developed. With the aid of this algorithm it is shown that the lower bound on S obtained in Sec. III is also an upper bound.

The play of a typical game is illustrated in Fig. 1. This example is discussed in some detail in the final section.

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*The game is trivial if $V_G \geq V_A$.

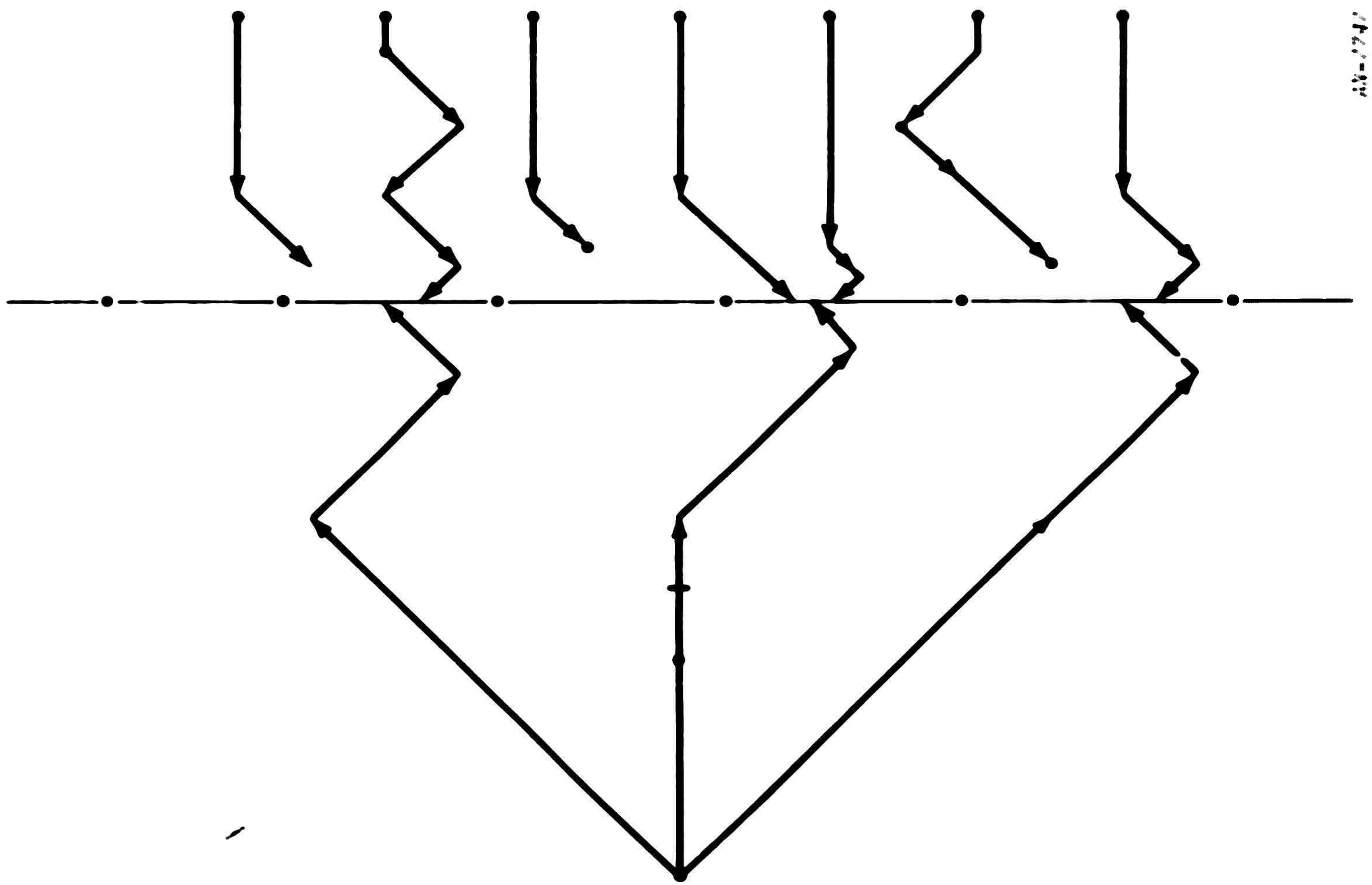

Figure 1. The Play of a Typical Game. This play of the game is described in Sec. VI. The diagram is drawn to scale with $v_G/v_A = 1/2$ and the forward velocity of each participant is equal to the maximum transverse velocity.

II. ONE ATTACKER

In this section we shall establish a notation for the subsequent discussion by briefly reviewing the results for the special case of one attacker against n goalies.*

If the attacker's position at time t is y(t) then he can reach the position

$$y^{max}(t) = y(t) + (T - t) V_A \tag{4}$$

at the end of the game by moving to the right for the remainder of the game. Similarly the attacker can reach the point

$$y^{min}(t) = y(t) - (T - t) V_A \tag{5}$$

by moving to the left for the remainder of the game. As indicated $y^{max}(t)$ and $y^{min}(t)$ are functions of the time and $\dot{y}^{max}(t) = 0$, $\dot{y}^{min}(t) = 2V_A$ if the attacker is moving to the right with velocity V_A. The quantities $y^{max}(t)$ and $y^{min}(t)$ bound the region under attack. The size of this region continually decreases.

Similarly the region defended by the jth goalie at time t is bounded by

$$x_j^{max}(t) = x_j(t) + V_G(T - t) \tag{6}$$

and

$$x_j^{min}(t) = x_j(t) - V_G(T - t) \tag{7}$$

*In Ref. 2, goal constraints were placed on the attacker. Such considerations are ignored throughout this paper.

For a given sized region under attack there is a minimum score corresponding to an optimal location of the goalies. This score is proportional to the size of the region under attack with a proportionality constant which depends on the number of goalies so that

$$S = (y_{max}(t) - y_{min}(t))/A_n \tag{8}$$

In Ref. 2 it was shown that

$$A_n = 2\alpha + 4\alpha^2 + \ldots + 2^\ell \alpha^\ell + 2(r + 1)\alpha^{\ell+1} \tag{9}$$

where $\alpha = V_A/(V_A - V_G)$ and $n = 2^\ell + r$ with $2^\ell > r \geq 0$.

An equation such as Eq. (8) holds only for an isolated instant in time. The attacker must move to the right (or the left) until he has eliminated roughly half of the goalies. (A goalie is eliminated from the game when $y^{min}(t) - x_j^{max}(t) = S$ or $x_j^{min}(t) - y^{max}(t) = S$). The attacker then finds himself attacking a much smaller region optimally guarded by a much smaller number of goalies.

This process is repeated until finally the attacker is evading only one goalie.

III. BASIC ATTACK THREAT

We consider m attackers optimally attacking a region of width $y_m^{max} - y_1^{min}$. The simplest example of such an optimal attack corresponds to each attacker having access to the entire region under attack. Throughout the rest of this section we will assume this to be the initial state. Given this attack there will be a best defense deployment of goalies yielding a miss distance S. This miss distance will be proportional to the size of the region under attack with a proportionality constant which is a function of n and m so that we may write

$$S = (y_m^{max} - y_1^{min})/A_n^m \tag{10}$$

It is instructive to consider the trivial case $m > n$ before going on to the more general situation. If there are more attackers than goalies the optimum strategies and the end result of the game are obvious. The goalies must plan to end the game as illustrated in Fig. 2. The attackers must plan to end the game with at least one attacker at each of the end-points of the region and at least one of the attackers at each of the mid-points of the line segments between the final positions of the goalies. It is then obvious that for $m > n$, $A_n^m = 2n$.

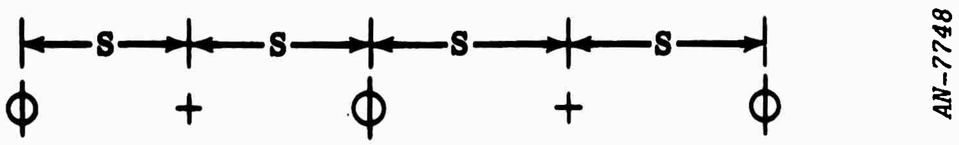

Figure 2. Optimal End Situation When $V_G = 0$ or $m > n$. The final positions of the two goalies and three attackers are indicated by crosses and circles respectively.

There are several features of this simple situation which are present also when $m < n$. It is first of all obvious that the initial optimal disposition of the goalies is not unique. The only requirement is that each goalie has access to his final position.

We shall say that a goalie has been pinned if his optimal strategy is to seek a final fixed position. A pinned goalie effectively divides the region under attack into two subregions and the attack assigns two subgroups to attack each of these regions. In the trivial case considered all the goalies have been pinned.

Pinning a goalie is a highly rewarding tactic for the offense. The velocity capability of a goalie who has been pinned early in the game is effectively nullified since a well-placed slow-moving goalie could do the job just as well. We shall therefore assume that the optimum offense is to pin as many goalies as early as is possible.

The pinning of a goalie is illustrated in Fig. 3. The attack threatens to send m' goalies into a region on the left and m - m' goalies

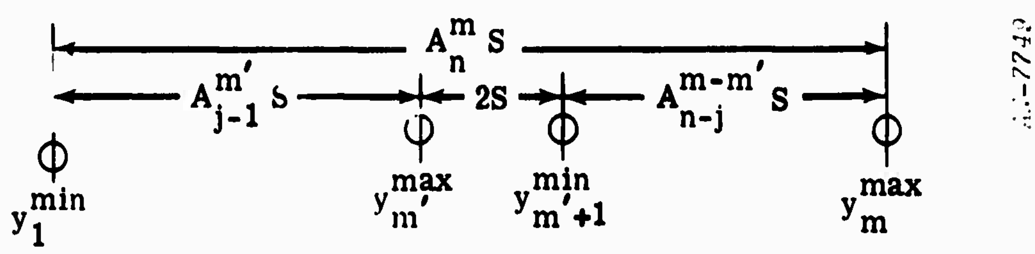

Figure 3. Attack Threat Giving Rise to a Pin.

into a region on the right.* In order to defend against such an attack the defense must commit $j - 1$ goalies to the left-hand region and $n - j$ goalies to the right-hand region. The jth goalie is pinned and must have access to the midpoint of the region of length 2S. If the jth goalie is too far to the right to have access to this midpoint the offense can achieve a larger miss distance than S by throwing as many attackers as he can (and this number is by postulate at least as large as m') into the region to the left of the jth goalie.

In general one must have the inequalities

$$A_n^m \leq (2 + A_{j-1}^{m'} + A_{n-j}^{m-m'}), \quad m > m' \geq 1 \tag{11}$$

The inequality sometimes holds since not all goalies are "pinnable" and not all partitions of the attackers into two groups are suitable for pinning a particular goalie. If the inequality were in the other direction the offense could obviously achieve a miss distance larger than S by committing m' goalies to the region of length $A_{j-1}^{m'}$ S on the left and the remaining goalies to a region larger than $A_{n-j}^{m-m'}$ S on the right. Even if the $n-j$ goalies are able to respond optimally they cannot prevent the attack from obtaining a score greater than S.

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*The existence of such a threat is sufficient, and it is not necessary that the attacker is able to begin the play in the two regions simultaneously. This is probably best appreciated by considering the play of a typical game as described in Sec. VI. When the number of goalies is sufficiently small a pinned goalie remains pinned throughout the play of the game. For a larger number of goalies the pin may be released and the pinned goalie may then inflict a smaller miss distance on one of the attackers. However, at least one of the attackers will get through with the calculated miss distance.

Since the right-hand side of Eq. (11) involves superscripts which are smaller than m it is possible to reason inductively to establish the following inequality*

$$A_n^m \leq 2(m - 1) + (m - u)A_\ell + uA_{\ell+1} \tag{12}$$

where $n - (m - 1) = m\ell + u$ and $m > u \geq 0$.

So far we have only established Eq. (12) with an inequality sign since there is the possibility that a more sophisticated attack might yield a larger miss distance (smaller A_n^m) than that implied by assuming an equality sign in Eq. (12). However, in Sec. V we shall show that the right-hand side of Eq. (12) is also a lower bound on A_n^m by considering defense strategies. Therefore, the equality sign holds in Eq. (12) and in our subsequent discussions we shall slightly anticipate Sec. V and generally assume the equality sign.

This formula implies that eventually m - 1 goalies get pinned and they each contribute 2 to the sum on the right-hand side of Eq. (12). The remaining goalies make a contribution of A_ℓ or $A_{\ell+1}$ each. A large A_n^m implies a small miss distance and since $A_\ell > 2$ for $\ell \geq 1$ the unpinned goalies are more effective than the pinned ones. From Eq. (9) it can be seen that the A_n are in arithmetic progression for $2^{k+1} - 1 \geq n > 2^k - 1$. As a result there are several alternate forms for A_n^m which are interesting consequences when one considers specific examples.

The constraints on the initial deployment for the optimal defense can be determined by considering an attack in which all the attackers go to the right or the left of the jth goalie. By inspecting Fig. 4 we find that

$$y_m^{max}(0) - x_j^{max}(0) \leq S + A_{n-j}^m S \tag{13}$$

*When m = 1 we drop the superscript.

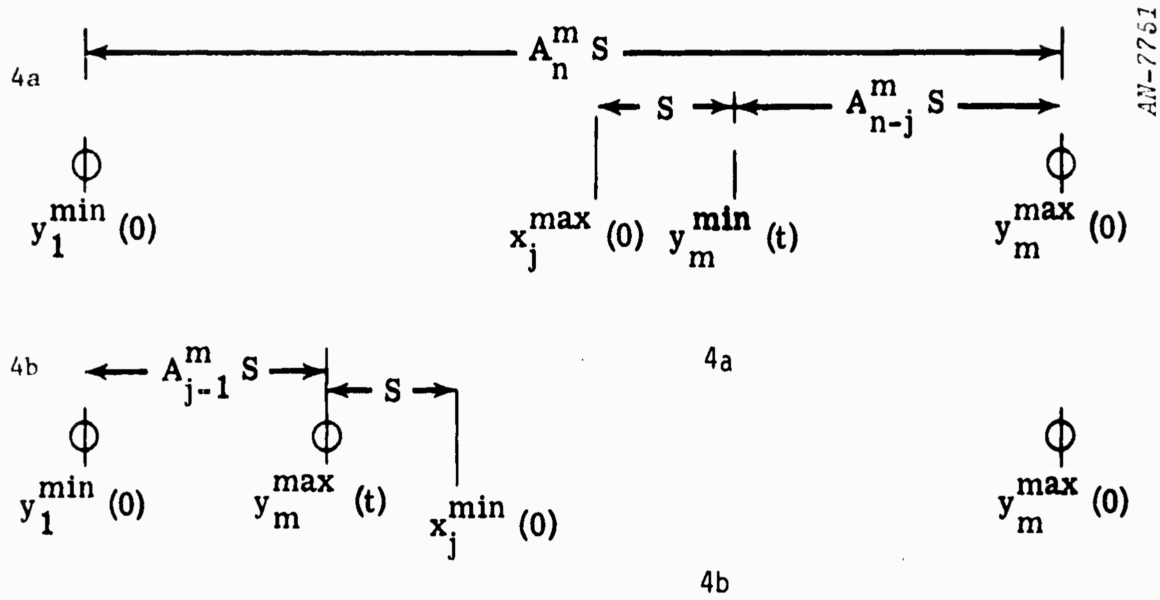

Figure 4. Determination of Constraints on Defense Deployment. In Fig. 4a all the attackers rush to the right of the jth goalie while they rush to his left in Fig. 4b.

and

$$x_j^{min}(0) - y_1^{min}(0) \leq S + A^m_{j-1} S \tag{14}$$

Since

$$x_j^{max}(0) - x_j^{min}(0) = \frac{\alpha - 1}{\alpha} (y_m^{max}(0) - y_m^{min}(0)) \tag{15}$$

these inequalities are consistent only if

$$A^m_n \leq \alpha(2 + A^m_{j-1} + A^m_{n-j}) \tag{16}$$

(Note that $(\alpha - 1)/\alpha = V_G/V_A$.) It is not difficult to show that this upper bound on A^m_n is less severe than that imposed by Eq. (12) except for $m = 1$. Therefore the equality sign can hold in Eq. (16) only for $m = 1$

and then only for certain values of j. In Ref. 2 it was shown that Eq. (16) (with $m = 1$) leads to the formula for A_n given in Eq. (9).

Therefore the initial position of the goalies is not uniquely prescribed (except for $m = 1$, when some but not all of the goalies must be at definite positions).

IV. BASIC DEFENSE

For the most part the defense against m attackers is the same as the defense against one attacker in that as the game proceeds individual goalies become exposed to the risk that some group of attackers will attack either on their right or left and they must respond appropriately. That is, with one attacker the attacker can make dynamic errors (heading in an inappropriate direction) which the defense can exploit but he cannot make a passive error (by being in the "wrong" place).

When there are several attackers there is the possibility that the attackers are initially inappropriately distributed over the region under attack. As the simplest example a subgroup of the attackers could all be very far to the right while the remaining attackers are very far to the left. The defense must then regard this attack as two independent attacks and assign a subgroup of the goalies to the leftmost group and the remaining goalies to the rightmost group. It is then possible that the miss distance will be less than $(y_m^{max} - y_1^{min})/A_n^m$.

The pinning mechanism may be used to the defense's advantage when the situation illustrated in Fig. 5 obtains. By assigning the jth goalie to the final position z_j the defense can force the m_1 goalies into a region S' units to the left of z_1 and the remaining $m - m_1$ goalies into a region S' units to the right of the position z_j provided

$$y_{m'}^{max} - z_j \leq S' \tag{17}$$

and

$$z_j - y_{m_j'+1}^{min} \leq S' \tag{18}$$

where S' is the miss distance to be determined. If the remaining goalies are suitably placed the defense can seek the score S' given by

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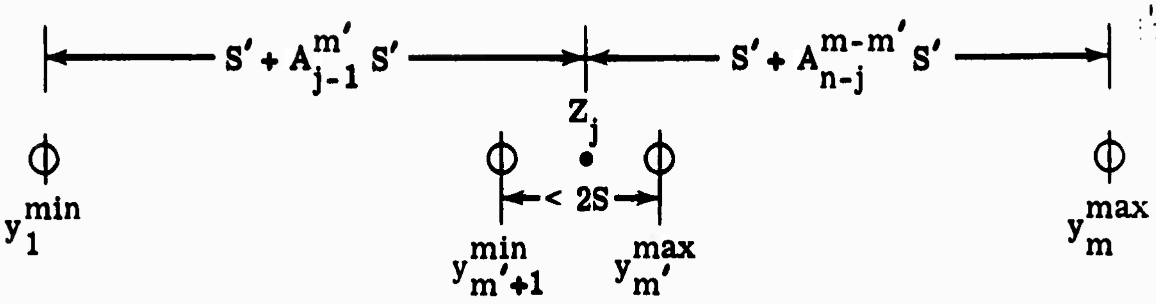

Figure 5. Situation Where Defense Resorts to a Pin.

$$S' = (z_j - y_1^{min})/(1 + A_{j-1}^{m'}) = (y_m^{max} - z_j)/(1 + A_{n-j}^{m-m'}) \quad (19)$$

provided the resulting z_j satisfies the inequalities (17) and (18). By eliminating z_j we find that

$$S' = (y_m^{max} - y_1^{min})/(2 + A_{j-1}^{m'} + A_{n-j}^{m-m'}) \quad (20)$$

If, for the particular values of m, m', n, and j, the inequality in Eq. (11) is a strong one then $S' < (y_m^{max} - y_1^{min})/A_n^m$ and the attackers' deployment was nonoptimal.

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V. ALGORITHM FOR STRATEGIES WITH ARBITRARY INITIAL DEPLOYMENT

Just as in the case of one attacker it is possible to develop an algorithm which determines the miss distance and the optimal strategies for an arbitrary initial deployment of the attackers and the goalies. Just as in the case for one attacker we develop the algorithm using A_n^m as given by Eq. (12) with the equality sign. If the true A_n^m is smaller than this value (if the defense is striving for a miss distance smaller than they can actually achieve) the defense will receive contradictory instructions. We shall prove that the algorithm we develop never gives rise to contradictory instructions to the defense and that the right-hand side of Eq. (12) is therefore also a lower bound on A_n^m. Hence, Eq. (12) holds with an equality sign.

The determination of the score function for the completely general case is quite simple in principle but quite cumbersome in practice. In order to illustrate the ideas we first go through the arguments for a trivial case--two attackers vs one goalie.

The score function is then obviously given by

$$S = MAX \Big[(y_2^{max} - y_1^{min})/B,\ MAX(y_2^{max} - x_1^{max},\ y_1^{max} - x_1^{max}),$$
$$MAX(x_1^{min} - y_1^{min},\ x_1^{min} - y_2^{min}) \Big] \tag{21}$$

with B = 2.

Now let us assume that an incorrect B is used in Eq. (21). For example, consider the standard attack and let $V_G/V_A > 1/2$, the goalie is initially somewhere near the center of the region under attack, and assume B = 4. The "score" will be given by

$$"S" = (y_2^{max}(t) - y_1^{min}(t))/4 \tag{22}$$

for some period of time. In order to prevent this "score" from decreasing the second attacker must move to the right and the first attacker to the left. No matter what the goalie does the "score" will ultimately take the form

$$"S" = (y_2^{max}(t) - y_1^{min}(t))/4 = (y_2^{max}(t) - x_1^{max}(t))$$

$$= (x_1^{min}(t) - y_1^{min}(t)) \qquad (23)$$

The second attacker moves to the right and the first attacker moves to the left in order to prevent the first term from decreasing. However the goalie must move to the right to prevent the second term from increasing and move to the left in order to prevent the third term from increasing. This contradiction informs everyone that the assumed value of B was too large. On the other hand if Eq. (21) is used with too small a value of B (say B = 1) the defense will never receive contradictory instructions (however, the true score at the end of the game need not bear any relationship to the assumed score through the course of the game). Therefore, Eq. (21) is a variational principle for B. If one takes an assumed value of B, call it 'B', then

contradictory instructions to defense can occur $\rightarrow$ 'B' > B

contradictory instructions to defense cannot occur $\rightarrow$ 'B' $\leq$ B .

Now let us return to the general case. We consider m attackers on n goalies. We tentatively assume that A_n^m is given by the right-hand side of Eq. (12). Our proof is inductive in character so it is legitimate to argue that $A_n^{m'}$ is indeed given by the right-hand side of Eq. (12) for all m' < m.

We introduce the quantity $S(m_1,m_2,n_1,n_2,U_1,U_2)$ which is the miss distance which can be achieved by the attackers from m_1 to m_2 attacking the region bounded by U_1 and U_2 which is defended by the goalies n_1 to n_2. Thus U_1 might be $y_{m_1}^{min}$, or $x_{n_1}^{max}$, or a pinned goalie's final position z_{n_1}.

The score is then determined by calculating the following table:

Case 1: Where the score function takes the form

$$\underline{S\left(m_1, m_2, n_1, n_2, y_{m_1}^{min}, y_{m_2}^{max}\right)}$$

A. Defense Options

Calculate:

$$\mathrm{MAX}\left[S\left(m_1, m_3, n_1, n_3, y_{m_1}^{min}, y_{m_3}^{max}\right), S\left(m_3 + 1, m_2, n_3 + 1, n_2, y_{m_3+1}^{min}, y_{m_2}^{max}\right)\right]$$

for all m_3 and n_3

Calculate:

$$\mathrm{MAX}\left[S\left(m_1, m_3, n_1, n_3, y_{m_1}^{min}, z_{n_3}\right), S\left(m_3 + 1, m_2, n_3, n_2, z_{n_3}, y_{m_2}^{max}\right)\right]$$

Where:

n_3 is chosen first to make the number calculated as small as possible, m_3 is then chosen to make the result as large as possible, and then z_{n_3} is adjusted to make the result as small as possible. This option cannot be employed by the defense unless $x_{n_3}^{max} > z_{n_3} > x_{n_3}^{min}$ for all possible choices of m_3.

Let S_A be the smallest number calculated in this table.

B. Attack Options

Calculate: $\left(y_{m_2}^{max} - y_{m_1}^{min}\right) / A_{n_2-n_1+1}^{m_2-m_1+1}$

Calculate: $S\left(m_1, m_2, n_3, n_2, x_{n_3}^{max}, y_{m_2}^{max}\right)$

Calculate: $S\left(m_1, m_2, n_1, n_3, y_{m_1}^{min}, x_{n_3}^{min}\right)$

Calculate: $S\left(m_1, m_2, n_3, n_4, x_{n_3}^{max}, x_{n_4}^{min}\right)$

For all n_3 and n_4 between n_1 and n_2 let S_B = largest number calculated.
Then $S\left(m_1, m_2, n_1, n_2, y_{m_1}^{min}, y_{m_2}^{max}\right) = \text{MIN}\left|S_A, S_B\right|$

Case 2: Where the score function takes the form

$$\underline{S\left(m_1, m_2, n_1, n_2, x_{n_1}^{max}, y_{m_2}^{max}\right)}$$

A. Defense Options

Calculate: $S\left(m_1, m_2, n_1, n_2, y_{m_1}^{min}, y_{m_2}^{max}\right)$

which is relevant if $y_{m_1}^{min} - x_{n_1}^{max}$ is too large

Calculate:

$$\text{MAX}\left[S\left(m_1, m_3, n_1, n_1, x_{n_1}^{max}, y_{m_3}^{max}\right), S\left(m_3 + 1, m_2, n_1, n_3, x_{n_1}^{max}, y_{n_2}^{max}\right)\right]$$

The attack has already committed himself to go to the right of the n_1th goalie. If ${}_{m_3}^{max} - x_{n_1}^{max}$ is too small the attackers from m_1 to m_3 are not really participating in the attack.

The remaining defense options are analogous to those given for $S\left(m_1, m_2, n_1, n_2, y_{m_1}^{min}, y_{m_2}^{max}\right)$. Let S_A be the smallest entry in this table.

B. Attack Options

Calculate: $$\left(y_{m_2}^{max} - x_{n_1}^{max}\right) \Big/ \left(1 + A_{n_2-n_1-1}^{m_2-m_1}\right)$$

The remaining attack options are the analogues of those previously given and S_B is the largest entry in this table. Again the score is $MIN\left[S_A, S_B\right]$.

The remaining tables are constructed in a similar fashion. The score for the game is $S(1,m,1,n,y_1^{min},y_m^{max})$. The score can be calculated in a finite number of steps, involving nothing more difficult than solving linear algebraic equations. The answer will be a numerical value for the score and a set of functional forms which give rise to the numerical value and dictate the optimal defense strategies.

Since we have used an upper bound for A_n^m the only possible difficulty is that A_n^m is too large. This will manifest itself in contradictory instructions to the defense. The author has examined such possibilities and ruled them out. For example, the score might be given by

$$S = \left(y_m^{max} - x_j^{max}\right) / \left(1 + A_{n-j}^{m_2+m_3}\right) = \left(x_j^{min} - x_k^{max}\right) / \left(2 + A_{j-k-1}^{m_1+m_2}\right) \qquad (24)$$

where the attack has the option of putting $m_2 + m_3$ attackers to the right of the jth goalie or $m_1 + m_2$ attackers between the jth and kth goalie.

The notation implies that m_1 attackers must go into one slot, m_3 must go into the other slot, but that m_2 attackers have the potential of invading either region. Thus $m_1 + m_2 + m_3 = m$. In order for this form to appear all the goalies except the jth goalie can optimally resist the attack. The jth goalie, however, has instructions to move to the left as well as to the right. The defense can try to get out of this dilemma by allowing the jth goalie to be pinned at z_j. The defense then seeks the score S' where

$$S' = \left(z_j - x_k^{max}\right) / \left(2 + A_{j-k-1}^{m_1+m_2'}\right) = \left(y_m^{max} - z_j\right) / \left(1 + A_{n-j}^{m_2''+m_3}\right)$$

$$= \left(y_m^{max} - x_k^{max}\right) / \left(3 + A_{j-k-1}^{m_1+m_2'} + A_{n-j}^{m_2''+m_3}\right) \tag{25}$$

where $m_2' + m_2'' = m_2$. The dilemma for the defense persists only if $S' > S$. We shall now show that $S' > S$ leads to a contradiction. We first note that it is a matter of simple algebra to show that $S' > S$ implies

$$x_j^{max} > z_j > x_j^{min} \tag{26}$$

where we also use the fact that A_n^m is a decreasing function of m.

Consider

$$S'' = \left(y_m^{max} - x_k^{max}\right) / \left(1 + A_{n-k}^{m_1+m_2+m_3}\right) \tag{27}$$

It is clear that $S'' \leq S$ since otherwise the attack would choose the option represented by Eq. (27). However, Eqs. (11), (25), and (27) imply that

$$S' \leq S'' \leq S \tag{28}$$

so that the assumption $S' > S$ leads to a contradiction. Therefore, the defense does not receive conflicting instructions and the expression we have used for A_n^m is a lower bound on A_n^m. Hence

$$A_n^m = 2(m - 1) + (m - n) A_\ell + nA_{\ell+1} \tag{29}$$

where $n - (m - 1) = m\ell + n$ and $m > n \geq 0$.

VI. PLAY OF A TYPICAL GAME

In the course of developing the theory of this game the author has found it very helpful to play the game, on paper, for specific numerical examples. Believing that the interested reader will also find such an exercise valuable we shall discuss the sortie illustrated in Fig. 1.

We consider three attackers vs seven goalies. From our formulas we find

$$A_7^3 = 4 + A_1 + 2A_2 = 4 + 2A_1 + A_3 \qquad (30)$$

The second form for A_7^3 arises because A_1, A_2, and A_3 are in arithmetic progression. The first form suggests that the attack can threaten to pin two goalies and then play three independent games where one attacker is engaging one goalie and the other attackers are engaging two goalies each. Similarly the second form suggests that the attack threatens to engage three goalies with one of the attackers and have the other two attackers each occupying the attention of one goalie.

Let $S = 1$ so that the region under attack is A_7^3 units. We assume the game begins with our standard attack, each attacker covering the whole region, and that the defense is optimally placed. Initially the score is given by $(y_3^{max} - y_1^{min})/A_7^3$ so that the third attacker must move to the right and the first must move to the left. The middle attacker and all the goalies receive no instructions, so they can do anything they choose. We shall assume the middle attacker moves straight ahead until the score function instructs him to do otherwise.

In general, an interesting situation occurs when each attacker is threatening A_3 units of territory. At this instant the second, fourth, and sixth goalies must have arrived at very definite positions. The remaining goalies must lie in certain regions. For example the second goalie must guard against an attack by the first attacker to his left as

well as guarding against an attack by all three attackers on his right. The algebraic identity illustrated in Fig. 6a implies that these conditions precisely define the second goalie's position. (Remember that $(\alpha - 1)/\alpha = V_G/V_A$.) Similarly the middle goalie must protect against the possibility of two attackers going to his right or his left. This fixes him in the center of the attacked region (see Fig. 6b). In order to exercise these options one of the outside attackers has to change direction. However, because the regions individually attacked by the attackers strongly overlap, the attack need not exercise these options and can continue on its original course.

The next interesting situation occurs when each attacker is attacking A_2 units of territory. By this time the regions under attack have sufficiently small overlap so that the defense can assign the outer two goalies to the outer attackers and the central three goalies to the middle attacker. This defines the positions of the first, second, sixth, and seventh goalies. However, the first and seventh goalies' positions are also fixed by other considerations since they have to guard against the threat in which all three attackers go between the 1st and 7th goalies (see Fig. 6c). Both points of view impose precisely the same requirements on the position of the goalies! The same situation prevails for the second and sixth goalies. (See Fig. 6d.)

Since all attackers are attacking the same sized regions and the middle attacker is confronted by three goalies he will achieve a smaller miss distance than the outside goalies. The middle attacker is a sacrifice. The middle goalie cannot handle him alone so that the third and fifth goalies must offer assistance.

In the illustration we allowed the middle attacker to play optimally in the subgame of one attacker vs three goalies. This is not necessary since the game has been set up in such a way that the attack is satisfied if at least one attacker gets through with the calculated miss distance.

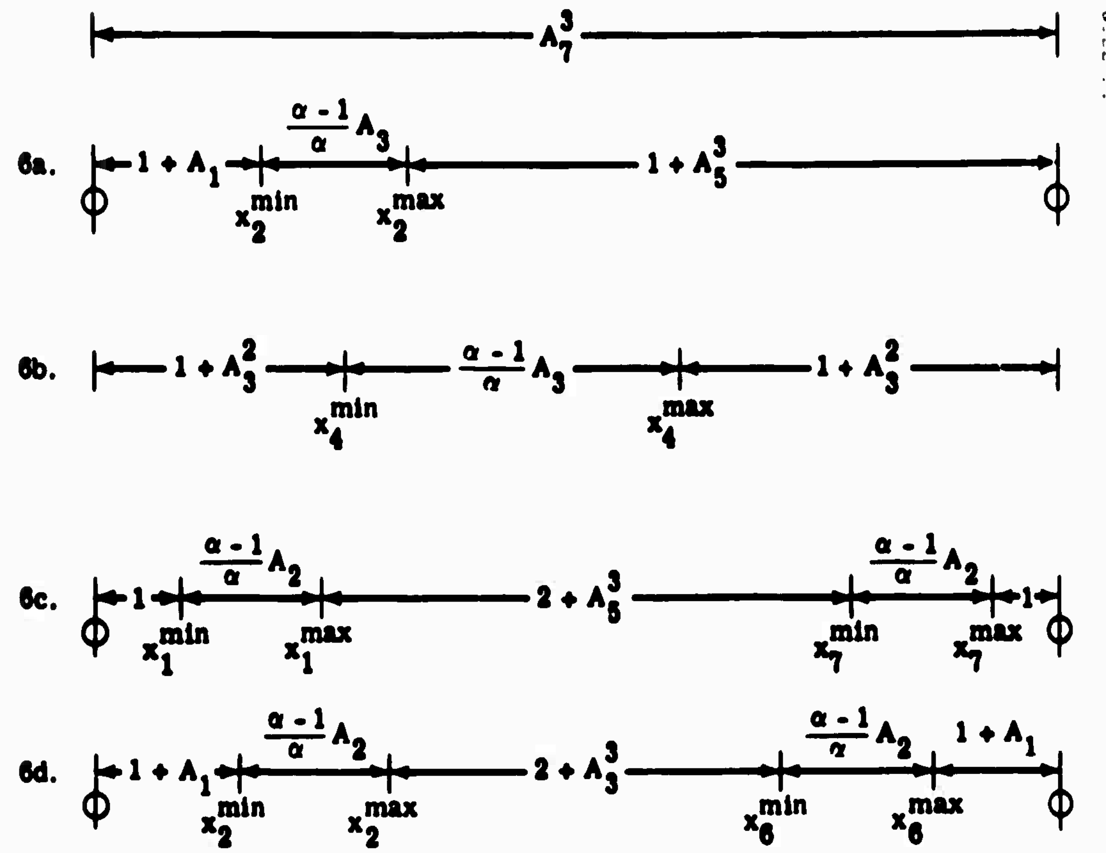

Figure 6. Critical Situations Occurring During the Course of the Game When Some of the Goalies Must Be at Precisely Defined Positions.

REFERENCES

1. N. Douglas, London Street Games, St. Catherines Press, London, 1916, p. 99.

2. P. J. Redmond, An Exactly Soluble Evasion Problem With an Arbitrary Number of Goalies, General Research, IMR-564, May 1967 (submitted to SIAM Journal on Control) (UNCLASSIFIED).